

Lecture 56

Complex Analysis

1. **Bounded set:** A set S is said to be bounded if for any $z \in S \subset \mathbb{C}$ there exists $K > 0$ such that $|z| \leq K$.
2. **Domain:** A domain S is an open connected set.
3. **Neighborhood of a point:** The neighborhood of a point $z_0 \in \mathbb{C}$ in the complex plane is the set of all points $z \in \mathbb{C}$ such that $|z - z_0| < \epsilon$, where $\epsilon > 0$ is arbitrary small.

$$Nbd(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon, \epsilon > 0 \text{ is arbitrary small}\}.$$

4. **Deleted neighborhood of a point:** A deleted neighborhood of $z_0 \in \mathbb{C}$ is defined by

$$\begin{aligned} Dnhd(z_0) &= Nbd(z_0) \setminus \{z_0\} \\ &= \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \setminus \{z_0\}. \end{aligned}$$

5. **Interior points:** A point $z_0 \in \mathbb{C}$ is said to be interior point of a set S if there exists an ϵ -neighborhood of z_0 such that $Nbd(z_0) \subset S$.
6. **Exterior points:** If every neighborhood of a point $z_0 \in \mathbb{C}$ contains points not belong to S , then z_0 is called exterior points.
7. **Boundary points:** If z_0 is neither interior nor exterior point, then it is called a boundary point.
8. **Open set:** A set S is said to be open if it contains only interior points.

Ex 1. $|z| < \ell$, Ex 2. $S = (1, 2) \rightarrow$ open set.

9. **Closed set:** A set S is said to be closed if it contains all its interior and boundary points.

Ex 1. $|z| \leq \ell$, Ex 2. $S = [1, 2] \rightarrow$ closed set, Ex 3. $S = \{z \in \mathbb{C} : |z - 1| \leq 5\}$.

10. **Function of a complex variable:** To each value of a complex variable $z \in \mathbb{C}$ if there corresponds one or more than one value of a complex variable ω , then we say ω is a complex function of z , i.e., $\omega = f(z)$.
11. **Limit of a complex function:** We say a number $\ell \in \mathbb{C}$ is a limit of a complex function $f(z)$ as z tends to $a \in \mathbb{C}$ and write $\lim_{z \rightarrow a} f(z) = \ell$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - \ell| < \epsilon, \quad \text{whenever} \quad |z - a| < \delta.$$

12. **Continuous function:** A complex function $f(z)$ is said to be continuous at $z = a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - f(a)| < \epsilon, \quad \text{whenever} \quad |z - a| < \delta.$$

13. **Derivative of a complex function:** If a function $f(z)$ is single valued in a domain D , then the derivative of f at $z = a$ in D is denoted by $f'(a)$ and it is defined by

$$f'(a) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a},$$

provided the limit exists.

14. **Analytic function:** Consider a single valued complex function $f(z)$ on a domain D . The function $f(z)$ is said to be analytic at $z = a$ if there exists a neighborhood of a such that $f'(z)$ exists for all $z \in Nbd(a)$, i.e., $f'(z)$ exists for all $z \in S = \{z \in \mathbb{C} : |z-a| < \delta\}$.

Differentiation at $z = a$ does not implies analytic at $z = a$.

Analytic at $z = a$ implies differentiation at $z = a$.

Lecture 57

15. **Singular point:** If $f'(z)$ exists at every point of a domain D except for a finite number of points. Then, $f(z)$ is said to be an analytic function on the domain D and those exceptional/non-analytic points are called singular points or singularities of f .

Ex. $f(z) = \frac{1}{z-2}$, 2 is a singular point.

In other words, a point $z_0 \in D$ is said to be a singular point if $f'(z_0)$ does not exists.

Theorem 1 (Cauchy-Riemann equation/CR-equation). *A complex function $f : \mathbb{C} \rightarrow \mathbb{C}$ is given by*

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$. The necessary condition that f is analytic on a domain D is that both u and v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{on } D.$$

If the partial derivatives of u and v are also continuous on D , then the CR-equation become sufficient condition for the f to be analytic on D .

Example. Verify whether the function $f : \mathbb{C} \rightarrow \mathbb{C}$ satisfies CR-equation or not at $z = 0$.

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

Solution: For $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$, and $v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

for $z \neq 0$. To verify the CR-equation, we need to only show that the relation also hold for $z = 0$. Now,

$$\begin{aligned} \left. \frac{\partial u}{\partial x} \right|_{(0,0)} &= \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^3}{x^3} = 1. \\ \left. \frac{\partial v}{\partial y} \right|_{(0,0)} &= \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{y^3}{y^3} = 1. \\ \left. \frac{\partial u}{\partial y} \right|_{(0,0)} &= \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{-y^3}{y^3} = -1. \\ \left. \frac{\partial v}{\partial x} \right|_{(0,0)} &= \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^3}{x^3} = 1. \end{aligned}$$

Hence, f satisfies CR-equation at $z = 0$. As, the partial derivatives at $z = 0$ are continuous at $z = 0$. The function f is analytic at $z = 0$.

Example. Find the analytic function $f = u + iv$, where $u(x, y) = e^x(x \cos y - y \sin y)$.

Solution: If f is analytic, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This implies

$$\frac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y, \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x(x \sin y - y \cos y) - e^x \sin y.$$

To determine v , we write

$$\begin{aligned} dv &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \\ &= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad (\text{by CR-equation}) \\ &= e^x(x \sin y - y \cos y + \sin y) dx + e^x(x \cos y - y \sin y + \cos y) dy \\ &= d(e^x x \sin y + e^x y \cos y) \\ v(x, y) &= e^x(x \sin y + y \cos y) + C \end{aligned}$$

The required analytic function:

$$f(z) = u + iv = e^x(x \cos y - y \sin y) + i[e^x(x \sin y + y \cos y) + C].$$

CR-equation in polar form: $x = r \cos \theta$, $y = r \sin \theta$. Then the CR-equation reduces to

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}.$$

Example. Verify whether $f(z) = z^m$, $m \in \mathbb{N}$ satisfies CR-equations.

Solution: $f(z) = z^m = r^m e^{im\theta} = \underbrace{r^m \cos m\theta}_u + i \underbrace{r^m \sin m\theta}_v$. Now,

$$\begin{aligned} \frac{\partial u}{\partial r} &= m r^{m-1} \cos m\theta = \frac{1}{r} \frac{\partial v}{\partial \theta}. \\ \frac{\partial u}{\partial \theta} &= -m r^m \sin m\theta = -r \frac{\partial v}{\partial r}. \end{aligned}$$

Therefore, $f(z) = z^m$ satisfies CR-equation.

Lecture 58

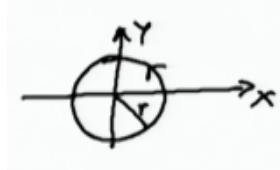
Complex integration: $\int_C f(z) dz$, where C is called the contour.

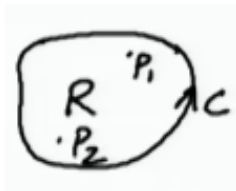
Contour: It means that C is composed of a continuous chain of finite number of regular arc.

Example. Let $f(z) = z^n$, $n \in \mathbb{N}$ and C is a closed curve $|z| = r$, $r > 0$. Evaluate $\int_C f(z) dz$.

Solution: The contour C is a circle of radius r centered at $(0, 0)$, i.e., $x^2 + y^2 = r^2$.

$$\begin{aligned} \int_C f(z) dz &= \int_C z^n dz \\ &= \int_0^{2\pi} r^n e^{in\theta} i r e^{i\theta} d\theta \\ &= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta = 0. \end{aligned}$$





Cauchy's Theorem: If a function f is analytic and single valued inside and on a simple closed curve or, contour C , then

$$\int_C f(z) dz = 0.$$

Example. Let $f(z) = \frac{1}{(z-1)^2}$. Then find the value of $\int_C f(z) dz$, where C is the curve/contour $|z - 1| = 1$.

Solution: Given contour is $|z - 1| = 1$ implies the circle of radius 1 centered at $(1, 0)$. Clearly, $f(z)$ is not analytic at $z = 1$ which does not belong to region R bounded by the curve C . In other words, $f(z)$ is always analytic in R . Therefore, by Cauchy's theorem,

$$\int_C f(z) dz = 0.$$

Cauchy's integral formula: If f is an analytic function within and on a closed curve C and if a is any point within C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz.$$

Higher order derivatives: If $f(z)$ is analytic within and on a closed contour C and a is any point within C , then the derivatives of all order are analytic and they are given by

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - a)^{n+1}} dz.$$

Lecture 59

Example. Let the given curve $C : |z| = 3$, then evaluate the integral $\int_C \frac{e^{2z}}{(z + 1)^4} dz$.

Solution: The given contour is the circle $x^2 + y^2 = 9$. Here $f(z) = \frac{e^{2z}}{(z+1)^4}$. Clearly, $z = -1$ lies within C . The function is not analytic at $z = -1$. Let $F(z) = e^{2z}$ and $n = 3$. Then by Cauchy integral formula of higher order

$$F^3(-1) = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z + 1)^4} dz.$$

$$F(z) = e^{2z}, \quad F'(z) = 2e^{2z}, \quad F''(z) = 4e^{2z}, \quad F'''(z) = 8e^{2z}.$$

Therefore, from the higher order Cauchy integral formula, we have

$$\int_C \frac{e^{2z}}{(z + 1)^4} dz = \frac{2\pi i}{6} \cdot 8e^{-2} = \frac{8\pi i}{3} e^{-2}.$$

Example. Let $C : |z| = 2$, then evaluate the integral $\int_C \frac{z}{(9 - z^2)(z + i)} dz$.

Solution: The given contour is the circle $x^2 + y^2 = 4$. Here $f(z) = \frac{z}{9 - z^2}$. At $z = -i$, the integrand is undefined/does not exist. We take $a = -i$, then by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz$$

$$\int_C \frac{z}{(9 - z^2)(z + i)} dz = 2\pi i f(a) = \frac{\pi}{5}.$$

Example. Evaluate the integral $\int_C \frac{dz}{z(z + \pi i)}$, where C is the contour given by $|z + 3i| = 1$.

Solution: The given contour is $|z + 3i| = 1$, i.e., the circle $x^2 + (y + 3)^2 = 1$. Let $f(z) = \frac{1}{z}$, and $a = -\pi i$. Then

$$\int_C \frac{f(z) dz}{(z + i\pi)} = 2\pi i f(-\pi i) = -2.$$

Example. Evaluate $I = \int_C z dz$, where C is the curve along the line $z = 0$ to $z = 1 + i$.

Solution:

$$I = \int_{z=0}^{1+i} z dz = \frac{(1 + i)^2}{2}.$$

Example. Evaluate $\int_C (x - y + ix^2) dz$, where C is along the line $z = 0$ to $z = 1 + i$.

Solution: Consider the path $y = x$, then $dy = dx$. Therefore, $dz = (1 + i) dx$. Hence

$$I = \int_0^1 (x - x + ix)(1 + i) dx = i(1 + i) \int_0^1 x^2 dx = \frac{i(1 + i)}{3}.$$

Lecture 60

Poles, Residues

Regular point and Singularities: A point $z = a$ is called a regular point for a complex-valued function f if f is analytic at a . A point $z = b$ is called a singular point or a singularity if f is not analytic at b but every neighborhood of b contains at least one point at which f is analytic.

A singular point b is said to be an isolated singular point if f is analytic in some deleted neighborhood of b . Otherwise, b is non-isolated singular point.

- (i) **Removable singularity:** An isolated singularity at $z = a$ of f is said to be a removable singularity if $\lim_{z \rightarrow a} f(z)$ exists in $\mathbb{C}$.

Example. $f(z) = \sin z$, $z \neq 1$. Then $z = 1$ is removable singularity and we define the function as

$$f(z) = \begin{cases} \sin z, & z \neq 1 \\ \sin 1, & z = 1. \end{cases}$$

- (ii) **Pole:** An isolated singularity $z = a$ of f is called a pole if $\lim_{z \rightarrow a} f(z) = \infty$.

- (iii) Essential singularity: An isolated singularity $z = a$ of f is called an essential singularity for f if $\lim_{z \rightarrow a} f(z)$ does not exist in $\mathbb{C}$.

Theorem 2. If f is analytic for $0 < |z - z_0| < R$ and has a pole of order m at z_0 , then

$$\begin{aligned} \text{Res}(f|z_0) &= \text{Res}(f : z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \Big|_{z=z_0} \\ &= \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]. \end{aligned}$$

Theorem 3 (Cauchy Residue Theorem). If f is analytic on and inside a simple closed curve C except at finitely many singular points $z_1, z_2, \dots, z_n$ then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f : z_k).$$

Example. Suppose $f(z) = \frac{\sin z}{(z^2-1)^2}$. Determine the order of the pole.

Solution: Here $f(z) = \frac{\sin z}{(z-1)^2(z+1)^2}$. Clearly, $z = 1$ and $z = -1$ are the poles of the function f .

Let us consider $z = 1$:

$$f(z) = \frac{g(z)}{(z-1)^2},$$

where $g(z) = \frac{\sin z}{(z+1)^2}$. Since g is analytic at 1 and $g(1) = \frac{\sin 1}{4} \neq 0$. We can conclude that $z = 1$ is a pole of order 2. Similarly, we can show that $z = -1$ is a pole of order 2.

Example. Determine the residue at $z_0 = 1$ of $f(z) = \frac{\sin z}{(z^2-1)^2}$ and $\int_C f(z) dz$, where $C = \{|z-1| = \frac{1}{2}\}$ is the circle radius $\frac{1}{2}$ and center at $(1, 0)$ oriented counter clockwise.

Solution: $z_0 = 1$ is a pole of order 2. We can also write $(z-1)f(z) = g(z) = \frac{\sin z}{(z+1)^2}$. Clearly, g is analytic at $z = 1$ with $g(1) \neq 0$. Then the residue of f at $z = 1$,

$$\begin{aligned} \text{Res}(f : 1) &= \frac{1}{(2-1)!} \lim_{z \rightarrow 1} \left[\frac{d}{dz} (z-1)^2 f(z) \right] \\ &= \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{\sin z}{(z+1)^2} \right] \\ &= \lim_{z \rightarrow 1} \frac{(z+1)^2 \cos z - 2(z+1) \sin z}{(z+1)^4} = \frac{\cos 1 - \sin 1}{4}. \end{aligned}$$

Then $\int_C f(z) dz = 2\pi i \text{Res}(f : 1) = \frac{\pi i}{2} (\cos 1 - \sin 1)$.